1.3 ONE-SIDED LIMITS

The functions we have considered so far have been defined by simple formulas, but there are many functions that cannot be described in this way Here are some examples: The population of Ottawa as a function of time; the cost of a taxi ride as a function of distance; the cost of mailing a first-class letter as a function of its mass. Such functions can be given by different formulas in different parts of their domains.

Consider the function f described by

$$f(x) = \begin{cases} x^2 & \text{if } x \leqslant 1 \\ 3 - x & \text{if } x > 1 \end{cases}$$

Remember that a function is a rule. For this particular function the rule is the following: First look at the value of x. If it happens that $x \leqslant 1$, then the value of $f(x)$ is x^2 On the other hand, if $x > 1$, then the value of $f(x)$ is $3 - x$. For instance, we compute $f(0)$, $f(1)$, and $f(2)$ as follows:

Since $0 \leqslant 1$, we have $f(0) = 0^2 = 0$.
Since $1 \leqslant 1$, we have $f(1) = 1^2 = 1$
Since $2 > 1$, we have $f(2) = 3 - 2 = 1$

We now investigate the limiting behaviour of $f(x)$ as x approaches 1

Approaching From the Left

$x < 1$	$f(x) = x^2$
0.9	0.81
0.99	0.980 1
0.999	0.998 001

Approaching From the Right

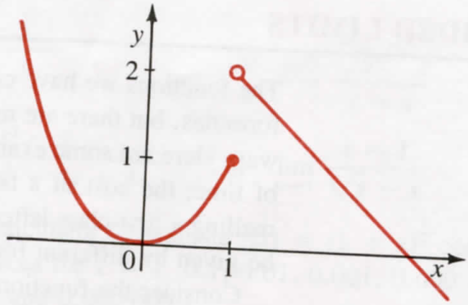
$x > 1$	$f(x) = 3 - x$
1 1	1.9
1.01	1.99
1.001	1.999

We see from the tables that $f(x)$ approaches 1 as x approaches 1 from the left, but $f(x)$ approaches 2 as x approaches 1 from the right. The notation we use to indicate this is

$$\lim_{x\to 1^-} f(x) = 1 \quad \text{and} \quad \lim_{x\to 1^+} f(x) = 2$$

Notice that the ordinary two-sided limit $\lim_{x\to 1} f(x)$ does not exist because the function approaches different values from the left and right.

Further insight into this type of function is gained from its graph. We observe that if $x \leqslant 1$, then $f(x) = x^2$, so the part of the graph of f that lies to the left of $x = 1$ must coincide with the graph of the parabola $y = x^2$ If $x > 1$, then $f(x) = 3 - x$, so the part of the graph of f that lies to the right of $x = 1$ coincides with the graph of $y = 3 - x$, which is a line with slope -1 The solid circle indicates that the point is included on the graph; the open circle indicates that the point is excluded from the graph.



In general, we write

$$\lim_{x \to a^-} f(x) = L$$

and say

"the **left-hand limit** of $f(x)$, as x approaches a, equals L"

or "the **limit of $f(x)$ as x approaches a from the left** equals L"

if the values of $f(x)$ can be made close to L by taking x close to a with $x < a$.

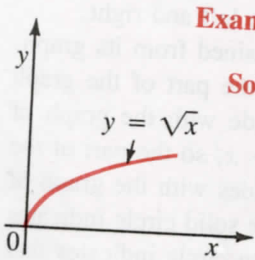
Similarly, if we consider only $x > a$, we have the **right-hand limit**:

$$\lim_{x \to a^+} f(x) = L$$

If a function has different expressions to the left and right of the number a, the following theorem provides a convenient way to test whether or not $\lim_{x \to a} f(x)$ exists.

> If $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, then $\lim_{x \to a} f(x)$ does not exist.
> If $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$, then $\lim_{x \to a} f(x) = L$.

When computing one-sided limits, we use the fact that the properties of limits listed in Section 1.2 also hold for one-sided limits.



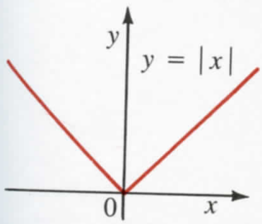
Example 1 Find $\lim_{x \to 0^+} \sqrt{x}$.

Solution Notice that the function $f(x) = \sqrt{x}$ is defined only for $x \geqslant 0$, so the two-sided limit $\lim_{x \to 0} \sqrt{x}$ does not make sense. If we let x approach 0 while restricting x to be positive, we see that $\sqrt{x}$ approaches 0:

$$\lim_{x \to 0^+} \sqrt{x} = \sqrt{\lim_{x \to 0^+} x} = \sqrt{0} = 0$$

Example 2 Show that $\lim_{x\to 0} |x| = 0$.

Solution Recall that



$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

Therefore $\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$

and $\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0$

Since the left and right limits are equal, we have

$$\lim_{x\to 0} |x| = 0$$

Example 3 The **Heaviside function** H is defined by

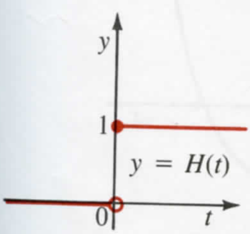
$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geqslant 0 \end{cases}$$

It is named after the electrical engineer Oliver Heaviside (1850–1925) and can be used to describe an electric current that is switched on at time $t = 0$. Evaluate, if possible,

(a) $\lim_{t\to 0^-} H(t)$ (b) $\lim_{t\to 0^+} H(t)$ (c) $\lim_{t\to 0} H(t)$

Solution (a) Since $H(t) = 0$ for $t < 0$, we have

$$\lim_{t\to 0^-} H(t) = \lim_{t\to 0^-} 0 = 0$$



(b) Since $H(t) = 1$ for $t > 0$, we have

$$\lim_{t\to 0^+} H(t) = \lim_{t\to 0^+} 1 = 1$$

(c) We see that $\lim_{t\to 0^-} H(t) \neq \lim_{t\to 0^+} H(t)$ and so $\lim_{t\to 0} H(t)$ does not exist.

Example 4 If

$$f(x) = \begin{cases} -x - 2 & \text{if } x \leqslant -1 \\ x & \text{if } -1 < x < 1 \\ x^2 - 2x & \text{if } x \geqslant 1 \end{cases}$$

determine whether or not $\lim_{x\to -1} f(x)$ and $\lim_{x\to 1} f(x)$ exist.

Solution We first compute the one-sided limits. Since $f(x) = -x - 2$ for $x < -1$, we have

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (-x - 2) = -(-1) - 2 = -1$$

Since $f(x) = x$ for $-1 < x < 1$, we have

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} x = -1$$

The left and right limits are equal, so

$$\lim_{x \to -1} f(x) = -1$$

Similarly, we have

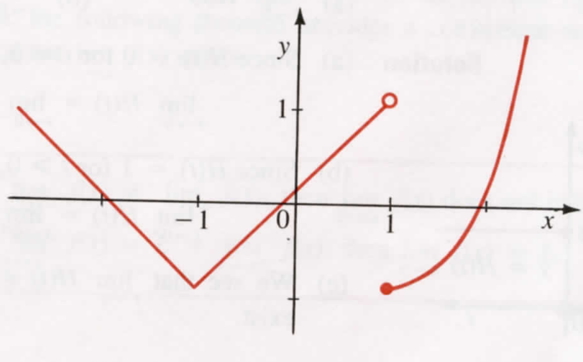
$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2 - 2x) = 1^2 - 2(1) = -1$$

The left and right limits are different, so

$$\lim_{x \to 1} f(x) \text{ does not exist}$$

This information is shown in the graph of f



Discontinuities

We recall from Section 1.2 the definition of a continuous function.

> f is **continuous** at a number a if
>
> $$\lim_{x \to a} f(x) = f(a)$$

Implicitly, this requires three things if f is continuous at a.

1 $f(a)$ is defined (so a is in the domain of f)

2. $\lim_{x \to a} f(x)$ exists

3. $\lim_{x \to a} f(x) = f(a)$

If f is not continuous at a, we say f is **discontinuous** at a, or f has a **discontinuity** at a.

For instance, the Heaviside function in Example 3 has a discontinuity at $t = 0$ because $\lim_{t \to 0} H(t)$ does not exist. Notice that there is a break in the graph of H at $t = 0$. This is typical of functions that have discontinuities. In fact, you can think of a continuous function as a function whose graph has no holes or breaks. You can draw its graph without removing your pencil from the paper Discontinuities occur where there are breaks in the graph.

Example 5 Where are the following functions discontinuous?

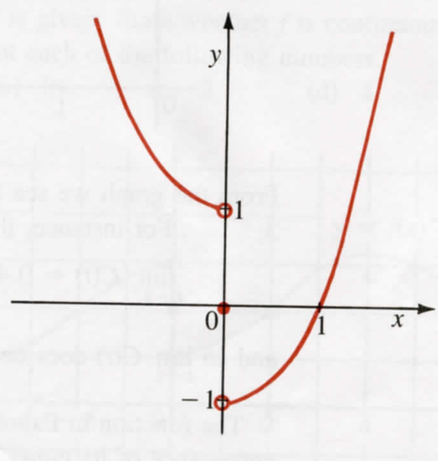
(a) $f(x) = \begin{cases} x^2 + 1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x^2 - 1 & \text{if } x > 0 \end{cases}$ (b) $g(x) = \begin{cases} x + 1 & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

Solution (a) When $x < 0$, we have $f(x) = x^2 + 1$, and we know polynomials are continuous. Similarly, $f(x) = x^2 - 1$ for $x > 0$. So f is continuous when $x \neq 0$. The only possibility for a discontinuity is $x = 0$, so we try to compute $\lim_{x \to 0} f(x)$.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x^2 + 1) = 0^2 + 1 = 1$$
$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x^2 - 1) = 0^2 - 1 = -1$$

Since the left and right limits are different, $\lim_{x \to 0} f(x)$ does not exist.

Therefore f is discontinuous at 0. This can also be seen from the break in the graph of f



(b) The only possibility for a discontinuity is $x = 2$. Since $g(x) = x + 1$ for $x \neq 2$, we have

$$\lim_{x \to 2} g(x) = \lim_{x \to 2} (x + 1) = 2 + 1 = 3$$

But, by definition, $\quad g(2) = 1$

So $\quad \lim_{x \to 2} g(x) \neq g(2)$

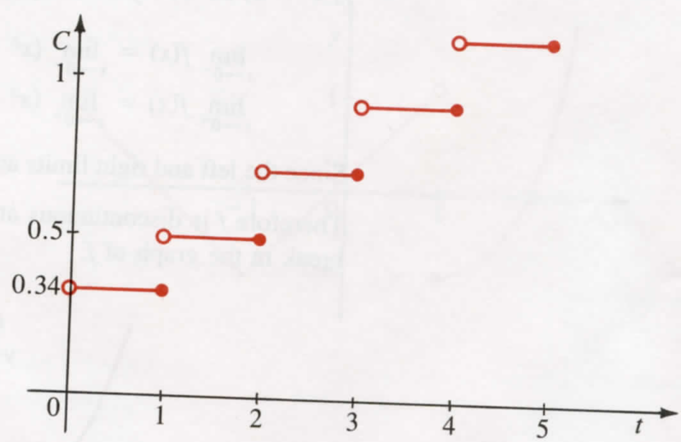
Therefore g is discontinuous at 2.

Example 6 The cost of a long-distance night-time phone call from Pine Bay to Hester is 26¢ for the first minute and 22¢ for each additional minute (or part of a minute). There is a minimum charge of 34¢ on all calls. Draw the graph of the cost C (in dollars) of a phone call as a function of the time t (in minutes). Where are the discontinuities of this function?

Solution From the given information, we have

$$\begin{aligned}
C(t) &= 0.34 && \text{if } 0 < t \leqslant 1 \\
C(t) &= 0.26 + 0.22 = 0.48 && \text{if } 1 < t \leqslant 2 \\
C(t) &= 0.26 + 2(0.22) = 0.70 && \text{if } 2 < t \leqslant 3 \\
C(t) &= 0.26 + 3(0.22) = 0.92 && \text{if } 3 < t \leqslant 4
\end{aligned}$$

and so on.



From the graph we see that there are discontinuities when $t = 1, 2, 3,$ For instance, the discontinuity at $t = 2$ occurs because

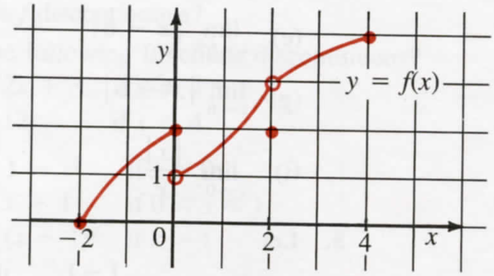
$$\lim_{t \to 2^-} C(t) = 0.48 \quad \text{and} \quad \lim_{t \to 2^+} C(t) = 0.70$$

and so $\lim_{t \to 2} C(t)$ does not exist.

The function in Example 6 is called a **step function** because of the appearance of its graph.

EXERCISE 1.3

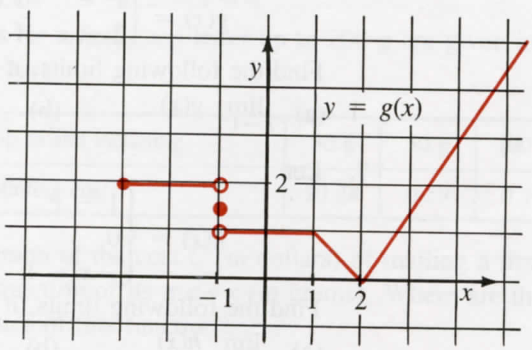
A 1. Use the given graph of f to state the value of the limit, if it exists.



(a) $\lim_{x\to -2^+} f(x)$ (b) $\lim_{x\to 0^-} f(x)$ (c) $\lim_{x\to 0^+} f(x)$ (d) $\lim_{x\to 0} f(x)$

(e) $\lim_{x\to 2^-} f(x)$ (f) $\lim_{x\to 2^+} f(x)$ (g) $\lim_{x\to 2} f(x)$ (h) $\lim_{x\to 4^-} f(x)$

2. Use the given graph of g to state the value of the limit, if it exists.

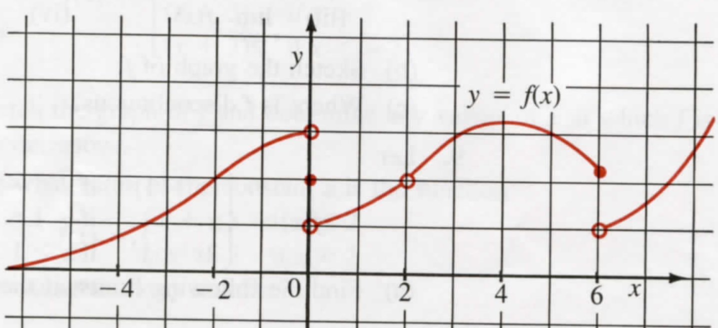


(a) $\lim_{x\to -3^+} g(x)$ (b) $\lim_{x\to -1^-} g(x)$ (c) $\lim_{x\to -1^+} g(x)$ (d) $\lim_{x\to -1} g(x)$

(e) $\lim_{x\to 2^-} g(x)$ (f) $\lim_{x\to 2^+} g(x)$ (g) $\lim_{x\to 2} g(x)$ (h) $\lim_{x\to 1} g(x)$

3. The graph of f is given. State whether f is continuous or discontinuous at each of the following numbers.

(a) -2 (b) 0 (c) 2 (d) 4 (e) 6



B **4.** Find the following limits, if they exist.

(a) $\lim_{x \to 0^+} \sqrt[4]{x}$ (b) $\lim_{x \to 3^+} \sqrt{x-3}$

(c) $\lim_{x \to 1^-} \sqrt{1-x}$ (d) $\lim_{x \to \frac{1}{2}^-} \sqrt[4]{1-2x}$

(e) $\lim_{x \to 6^+} |x-6|$ (f) $\lim_{x \to 6^-} |x-6|$

(g) $\lim_{x \to 6} |x-6|$ (h) $\lim_{x \to 0^+} \frac{|x|}{x}$

(i) $\lim_{x \to 0^-} \frac{|x|}{x}$ (j) $\lim_{x \to 0} \frac{|x|}{x}$

5. Let

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ x+1 & \text{if } x \geqslant 0 \end{cases}$$

Find the following limits, if they exist. Then sketch the graph of f

(a) $\lim_{x \to 0^-} f(x)$ (b) $\lim_{x \to 0^+} f(x)$ (c) $\lim_{x \to 0} f(x)$

6. Let

$$g(x) = \begin{cases} x^2 & \text{if } x \leqslant 1 \\ 2-x & \text{if } x > 1 \end{cases}$$

Find the following limits, if they exist. Then sketch the graph of g

(a) $\lim_{x \to 1^-} g(x)$ (b) $\lim_{x \to 1^+} g(x)$ (c) $\lim_{x \to 1} g(x)$

7. Let

$$h(x) = \begin{cases} 1-x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ -x-1 & \text{if } x > 0 \end{cases}$$

Find the following limits, if they exist. Then sketch the graph of h.

(a) $\lim_{x \to 0^-} h(x)$ (b) $\lim_{x \to 0^+} h(x)$ (c) $\lim_{x \to 0} h(x)$

8. Let

$$f(x) = \begin{cases} -1 & \text{if } x \leqslant -2 \\ \frac{1}{2}x & \text{if } -2 < x < 2 \\ 1 & \text{if } x \geqslant 2 \end{cases}$$

(a) Find the following limits.

(i) $\lim_{x \to -2^-} f(x)$ (ii) $\lim_{x \to -2^+} f(x)$

(iii) $\lim_{x \to 2^-} f(x)$ (iv) $\lim_{x \to 2^+} f(x)$

(b) Sketch the graph of f

(c) Where is f discontinuous?

9. Let

$$f(x) = \begin{cases} (x+1)^2 & \text{if } x < -1 \\ x & \text{if } -1 \leqslant x \leqslant 1 \\ 2x - x^2 & \text{if } x > 1 \end{cases}$$

(a) Find the following limits, if they exist.

(i) $\lim_{x \to -1^-} f(x)$ (ii) $\lim_{x \to -1^+} f(x)$ (iii) $\lim_{x \to -1} f(x)$

(iv) $\lim_{x \to 1^-} f(x)$ (v) $\lim_{x \to 1^+} f(x)$ (vi) $\lim_{x \to 1} f(x)$

(b) Sketch the graph of f

(c) Where is f discontinuous?

10. Where are the following functions discontinuous?

(a) $f(x) = \begin{cases} 2x + 3 & \text{if } x \neq 4 \\ 12 & \text{if } x = 4 \end{cases}$

(b) $f(x) = \begin{cases} 1 - x^2 & \text{if } x \leqslant 0 \\ x + 1 & \text{if } 0 < x \leqslant 1 \\ (x - 1)^2 & \text{if } x > 1 \end{cases}$

(c) $f(x) = \begin{cases} -x & \text{if } x < -1 \\ x^3 & \text{if } -1 \leqslant x \leqslant 1 \\ x & \text{if } x > 1 \end{cases}$

(d) $f(x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant 1 \\ x - 2 & 1 < x < 3 \\ x - 4 & \text{if } 3 \leqslant x \leqslant 4 \end{cases}$

11. Postal rates for a first-class letter up to 200 g are given in the following chart.

Up to and including	30 g	50 g	100 g	200 g
Mailing cost	\$0.38	0.59	0.76	1 14

Draw the graph of the cost C (in dollars) of mailing a first-class letter as a function of its mass x (in grams). Where are the discontinuities of this function?

12. A taxi company charges \$1.00 for the first 0.2 km (or part) and \$0.10 for each additional 0.1 km (or part). Draw the graph of the cost C of a taxi ride, in dollars, as a function of the distance travelled x (in kilometres). Where are the discontinuities of this function?

C **13.** Let

$$f(x) = \begin{cases} 1 - |x| & \text{if } |x| \leqslant 1 \\ |x| - 1 & \text{if } 1 < |x| \leqslant 2 \\ (x - 3)^2 & \text{if } x > 2 \\ (x + 3)^2 & \text{if } x < -2 \end{cases}$$

Sketch the graph of f and determine any values of x at which f is discontinuous.

14. For what value of the constant c is the function

$$f(x) = \begin{cases} x + c & \text{if } x < 2 \\ cx^2 + 1 & \text{if } x \geqslant 2 \end{cases}$$

continuous at every number?